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Entanglement and Optimal Dense Coding in Spin Chain with an Arbitrary Magnetic Field

Hailin Huang

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Abstract The entanglement and optimal dense coding at entangled states of a 1D Ising chain in the presence of an external magnetic field with an arbitrary direction, are investigated. The entanglement concurrence and the optimal dense coding capacity are calculated for different orientations of the magnetic field. It has been found that the direction of external magnetic field has effects on the entanglement concurrence and optimal dense coding capacity. In the case of antiferromagnet, the quantum phase transition occurs when an external magnetic field is parallel to Ising orientation. The concurrence increases when the angle between the direction of magnetic field and Ising orientation become smaller at ground state in certain parameter regimes, so does the optimal dense coding. The maximum moves toward the direction perpendicular to the Ising orientation in higher temperature. In contrast, the more concurrence and optimal dense coding can be produced only in the case of an external magnetic field perpendicular to Ising orientation at zero and low temperature for ferromagnetic case.

Keywords Entanglement · Optimal dense coding · Two-qubit chain

1 Introduction

Entanglement, a quantum mechanical property with no classical analog, plays an essential role in the many of the most exciting applications of quantum computation and quantum information [1, 2]. Quantum entanglement has been gradually regarded as a significant resource for super-dense cording [3], quantum teleportation [4, 5], and so on. Recently, quantum spins chain in condensed matter physics have been extensively investigated on characterizing entanglement, optimal dense coding and teleportation [6–10]. In thermal equilibrium at temperature T in the presence of an external magnetic field B along the z axis, authors have analyzed the influences of anisotropic interactions and a magnetic field B on

H. Huang (✉)
School of Physics, Huazhong University of Science and Technology, Wuhan 430074, China
e-mail: hailin_huang@whpu.edu.cn

the entanglement of a two-qubit anisotropic Heisenberg XY chain [11]. Yao et al. have studied the entanglement for low-dimensional quantum spin chains at finite temperature [12] and entanglement entropy for Hubbard model with hole-doping and external magnetic field [13], as well as the relationship between the entanglement and the quantum phase transition [14]. Zhang et al. have investigated the thermal entanglement in a two-qubit Heisenberg XXZ spin chain under an inhomogeneous magnetic field [15]. The separability conditions and limit temperatures of entanglement detection for two-qubit Heisenberg XYZ chain in the presence of a magnetic field are examined by N. Canosa et al. [16]. The dense coding with a general mixed state on the Hilbert space shared between a sender and receiver was studied in [17]. The author has also proved that the capacity of dense coding is maximized if the sender prepares the signal states by mutually orthogonal unitary transformations with equal a priori probabilities. Zhang [18] has discussed the effects of two kinds of anisotropy on the optimal dense coding in the anisotropic XXZ model and the Heisenberg model with DM interaction. The effects of anisotropy on the quantum entanglement and teleportation in a two-qubit Heisenberg XY chain with an external magnetic field were investigated by Yeo et al. [19]. The influences of spin-orbit coupling on the entanglement and teleportation were discussed for a two-qubit Heisenberg XXX chain in the absence of a magnetic field [20] and a two-qubit Heisenberg XYZ chain with spin-orbit interaction in an inhomogeneous magnetic field [21]. However, the optimal dense coding and entanglement concurrence for a two-qubit Heisenberg chain in a magnetic field with arbitrary direction has rarely been analyzed. These are the motivation of this Brief Report.

The one-dimension Ising model describing a set of linearly arranged spins is absent of entanglement completely. However, a component of an external magnetic field, along the direction perpendicular to the Ising orientation, no matter how small, is enough to produce entanglement in the antiferromagnetic case [22]. In this paper, both antiferromagnetic case and ferromagnetic case, the entanglement concurrence and optimal dense coding in a two-qubit Ising chain with an arbitrary direction magnetic field have been studied. It has been found that the entanglement concurrence and optimal dense coding capacity have consanguineous relevancy with the angle between the magnetic field and the Ising orientation. The entanglement behavior in the case of ferromagnet is not the same to that in antiferromagnetic case.

This paper is organized as follows. In Sect. 2, The Hamiltonian of the system is introduced, and the corresponding thermal density matrix is obtained. The entanglement at the ground state and excited states are discussed in Sect. 3. In Sect. 4, the optimal dense coding with entangled states is investigated. Finally, the conclusions are drawn in Sect. 5.

2 The Hamiltonian and Thermal Density Matrix

Considering the Hamiltonian H for the two-qubit Heisenberg Ising chain in an external magnetic field B with an arbitrary direction is [22]

$$H = 2J\sigma_1^z \otimes \sigma_2^z + B\sin\theta(\sigma_1^x \otimes I_2 + I_1 \otimes \sigma_2^x) + B\cos\theta(\sigma_1^z \otimes I_2 + I_1 \otimes \sigma_2^z), \quad (1)$$

where σ_α^z are the Pauli matrices at sites $\alpha = 1, 2$. The chain is said to be antiferromagnetic for $J > 0$ and ferromagnetic for $J < 0$. θ $(0 \le \theta \le \pi)$ is the angle between the direction of an external magnetic field and the Ising orientation. It is reasonable to confining the variation of B $(B \ge 0)$ within the plane containing the Ising orientation, because the Hamiltonian possesses rotational symmetry about the z axis in 3 spatial dimensions.

In the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the Hamiltonian has the following matrix form:

$$H = \begin{pmatrix} 2(J+B_z) & B_x & B_x & 0 \\ B_x & -2J & 0 & B_x \\ B_x & 0 & -2J & B_x \\ 0 & B_x & B_x & 2(J-B_z) \end{pmatrix}, \tag{2}$$

where defining $B_x := B\sin\theta$, $B_z := B\cos\theta$.

After straightforward calculation, the spectrum of H is obtained as

$$H|\psi_i\rangle = \varepsilon_i|\psi_i\rangle, \tag{3}$$

where the eigenstates and the corresponding eigenvalues are, respectively,

$$\begin{aligned} |\psi^i\rangle &= \frac{1}{N_i}(2B_x|00\rangle + (\varepsilon_i - 2J + 2B_z)|01\rangle + (\varepsilon_i - 2J + 2B_z)|10\rangle \\ &\quad + [(\varepsilon_i + 2J)(\varepsilon_i - 2J + 2B_z) - 2B_x^2]|11\rangle) \quad (i = 1, 2, 3), \\ |\psi^4\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \end{aligned} \tag{4}$$

and

$$\begin{aligned} \varepsilon_i &= 2\sqrt{-\frac{p}{3}}\cos\frac{\delta + (i-1)2\pi}{3} + \frac{2J}{3} \quad (i = 1, 2, 3), \\ \varepsilon_4 &= -2J. \end{aligned} \tag{5}$$

In above equations $N_i = \sqrt{4B_x^2 + 2(\varepsilon_i - 2J + 2B_z)^2 + [(\varepsilon_i + 2J)(\varepsilon_i - 2J + 2B_z) - 2B_x^2]^2}$ $(i = 1, 2, 3)$ is the normalization, defining $p := -4(\frac{4J^2}{3} + B^2)$, $q := -\frac{16}{27}J^3 + \frac{8J}{3}(2J^2 + 2B_x^2 - 4B_z^2)$ and $\delta = \arccos(-\frac{q}{2\sqrt{-(p/3)^3}})$.

In thermal equilibrium at temperature T, the state of above system is described by the density operator

$$\rho = \frac{1}{Z}[e^{-\beta\varepsilon_1}|\psi^1\rangle\langle\psi^1| + e^{-\beta\varepsilon_2}|\psi^2\rangle\langle\psi^2| + e^{-\beta\varepsilon_3}|\psi^3\rangle\langle\psi^3| + e^{-\beta\varepsilon_4}|\psi^4\rangle\langle\psi^4|], \tag{6}$$

where the partition function $Z = e^{-\beta\varepsilon_1} + e^{-\beta\varepsilon_2} + e^{-\beta\varepsilon_3} + e^{-\beta\varepsilon_4}$, the Boltzmann's constant $k \equiv 1$ for simplicity from hereon, and $\beta = 1/T$.

3 The Quantum and Thermal Entanglement

To measure the entanglement of the above system, the concurrence is considered. $C(\rho) = \max\{0, 2\lambda_{\max} - \sum_{i=1}^4 \lambda_i\}$, where the λ_i's are positive square roots of the eigenvalues of the non-Hermitian spin-flipped matrix operator $R = \rho(\sigma^y \otimes \sigma^y)\rho^*(\sigma^y \otimes \sigma^y)$, the asterisk indicates the complex conjugation. The value of $C(\rho)$ ranges from 0 (state with no entanglement) to 1 (state with maximum entanglement). In the case that the system is in the pure state, i.e. $\rho = |\psi\rangle\langle\psi|$, $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$, the above formula is simply written as $C(|\psi\rangle) = 2|ad - bc|$.

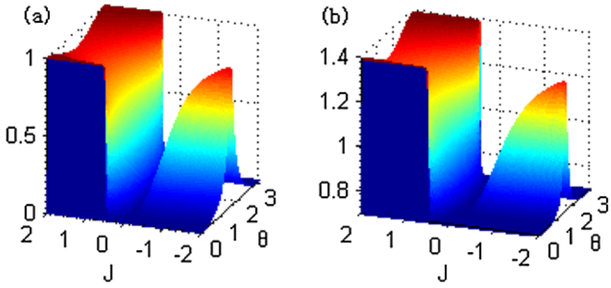

Fig. 1 (Color online) At zero temperature, the concurrence C (*on the left*) and the optimal dense coding χ (*on the right*) vs. the coupling constant J and the angle θ (scaled in radian) between the direction of an external magnetic field and the Ising orientation. We work in units where J and B are dimensionless

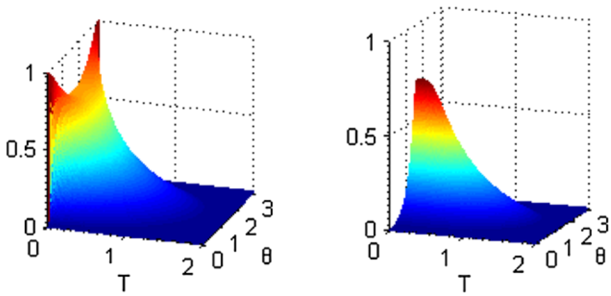

Fig. 2 (Color online) The concurrence C vs. temperature T and the angle θ (scaled in radian) between the direction of an external magnetic field and the Ising orientation. The *left* and *right figures* correspond to the antiferromagnetic case and ferromagnetic case, respectively. We work in units where J and B are dimensionless

Although the expression for the concurrence $C(\rho)$ can be solvable analytically, the expression is complicated, so the $C(\rho)$ is present in graphical form. At zero temperature, Fig. 1(a) shows the numerical solution when the concurrence is plotted as a function of J and θ. Obviously, the antiferromagnetic case and ferromagnetic case present different entanglement behavior. Figures are symmetric about $\theta = \pi/2$. For convenience, considerations here are restricted to $0 \le \theta \le \pi/2$. For antiferromagnetic case, in the region of $J > 0.5B$, the entanglement decreases with the increasing of θ, and reaches minimum at $\theta = \pi/2$. When the angle θ has exactly the value of 0, the entanglement vanishes. While the angle θ approaches to 0, i.e., in the limit $\theta \to 0$, the entanglement tends to 1 (the maximal entanglement). This indicates that the quantum phase transition has taken place at critical point $\theta = 0$ (i.e., $B_x = 0$). It is not difficult to understand. For $\theta = 0$ and $J > 0.5B$, an external magnetic field is parallel to Ising orientation. The ground state of the system is always the two-level energy ($\varepsilon_2 = \varepsilon_4 = -2J$) equiprobable degenerate Bell state $|\psi^{\pm}\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$, so there is no entanglement. However, for non-vanish angle θ, the ground state of the system is always higher entangled state $|\psi^2\rangle$. In the limit $\theta \to 0$, the energy level ε_2 of the system is close to the energy $-2J$, and the corresponding state $|\psi^2\rangle$ is close to the Bell states $|\psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. Therefore the maximal entanglement has achieved. Next, in the region of $J < 0.5B$, for small angle θ, although the ground state of the system is still state $|\psi^2\rangle$, it is non-entangled state $|11\rangle$ this time. For the ferromagnetic case, the entanglement increases monotonously with the increasing of θ. The maximal entanglement can be obtained at the point $\theta = \pi/2$. In one words, for antiferromagnetic case, in order to produce more entanglement in broad region of J, the angle between the direction of an external magnetic field and Ising orientation should be smaller as possible. However, in contrast, for ferromagnetic case, if and only if the angle between the direction of an external magnetic field and Ising orientation takes maximum, i.e., an external magnetic field is perpendicular to Ising orientation, more entanglement can be produced.

At nonzero temperature, Fig. 2 demonstrates the numerical solution when the concurrence is plotted as a function of T and θ. Due to mixing of excited states, the concurrence

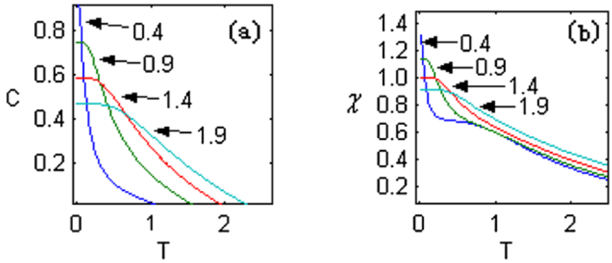

Fig. 3 (Color online) The concurrence C (**a**) and optimal dense coding χ (**b**) as a function of temperature T for different external magnetic fields B (from *top* to *bottom*, $B = 0.4, 0.9, 1.4, 1.9$), $J = 1$, $\theta = \pi/2$. We work in units where J and B are dimensionless

$C(\rho)$ decreases monotonously to zero as the temperature is increased beyond the critical value T_{c_1}, which is the function of B and θ. For the fixing magnitude of B, the critical temperature T_{c_1} increases with the increasing of θ for both antiferromagnetic case and ferromagnetic case. T_{c_1} reaches maximum at $\theta = \pi/2$. It indicates that more entanglement can be obtained when an external magnetic field is perpendicular to Ising orientation at higher temperature. It is also found that the critical temperature T_{c_1} for antiferromagnetic case is less than one for ferromagnetic case, while the concurrence has the same thermal behavior for both antiferromagnetic case and ferromagnetic case, when an external magnetic field is perpendicular to Ising orientation. Figure 3(a) shows that the critical temperature T_{c_1} increases with the increasing of external magnetic field B.

4 The Optimal Dense Coding with Entangled States

For considering the thermal entangled states of the two-qubit system as a channel, in order to carry out the optimal dense coding, the set of mutually orthogonal unitary transformations is necessary to be made. The set of mutually orthogonal unitary transformations of the optimal dense coding for two-qubit is given by [17]

$$\begin{aligned} &U_{00}|n\rangle = |n\rangle, \qquad U_{01}|n\rangle = |n+1 \ (\mathrm{mod}\ 2)\rangle, \\ &U_{10}|n\rangle = e^{\sqrt{-1}(2\pi/2)j}|n\rangle, \qquad U_{11}|n\rangle = e^{\sqrt{-1}(2\pi/2)j}|n+1 \ (\mathrm{mod}\ 2)\rangle, \end{aligned} \tag{7}$$

where $|n\rangle$ is the single qubit computational basis ($|n\rangle = |0\rangle, |1\rangle$). According to the unitary transformations (7), the average state of the ensemble of signal states generated is

$$\bar{\rho}^* = \frac{1}{4}\sum_{i=0}^{3}(U_i \otimes I_2)\rho(U_i^+ \otimes I_2), \tag{8}$$

where 0 stands for 00, 1 for 01, 2 for 10, 3 for 11, and ρ is the thermal states of the system.

After making the set of mutually orthogonal unitary transformations, the maximum dense coding capacity χ can be obtained by

$$\chi = S(\bar{\rho}^*) - S(\rho). \tag{9}$$

In above formula, $S(\bar{\rho}^*)$ is the von Neumann entropy of the average state of the ensemble of signal states $\bar{\rho}^*$, and $S(\rho)$ is the von Neumann entropy of the thermal state ρ.

Similarly, the expression for the maximum dense coding capacity of the entanglement channel is complicated, so the illustration for χ is given. In the following two paragraphs, two cases at different temperatures will be discussed explicitly.

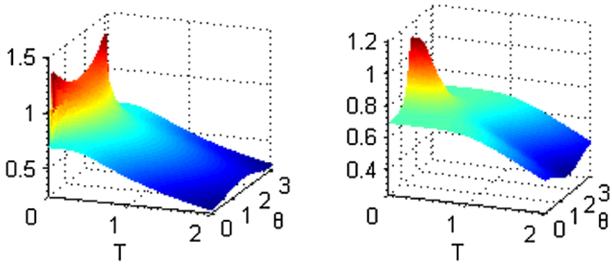

Fig. 4 (Color online) The optimal dense coding χ vs. temperature T and the angle θ (scaled in radian) between the direction of an external magnetic field and the Ising orientation. The *left* and *right figures* correspond to the antiferromagnetic case and ferromagnetic case, respectively. We work in units where J and B are dimensionless

First, at zero temperature, Fig. 1(b) shows the numerical solution of the maximal dense coding capacity as a function of J and θ. Compared with Fig. 1(a), it is not difficult to find that the optimal dense coding capacity exhibits analogous characteristic to entanglement, no matter what antiferromagnetic case and ferromagnetic case. It shows that the stronger entanglement is, the stronger dense coding capacity is. In order to be valid for optimal dense coding, for ferromagnetic case, there exists an angle θ_c, which is a function of J. If $\theta_c < \theta < \pi - \theta_c$, the quantum channel is valid for dense coding. For antiferromagnetic case, there exists a critical value J_c, which is a function of θ. If $J < J_c$, the optimal dense coding is infeasible through the quantum channel.

Next, at finite temperature, Fig. 4 demonstrates the numerical solution of the optimal dense coding capacity. The amount that the maximal dense coding capacity can reach reduces monotonously with the increasing of temperature. There exists a critical temperature T_{c_2}, beyond which the optimal dense coding is infeasible. The critical temperature T_{c_2} is a function of B and θ. At low temperature, the area around $\theta = 0$ (or $\theta = \pi$) for antiferromagnetic case, $\theta = \pi/2$ for ferromagnetic case, is useful to optimal dense coding. Similar to concurrence, the optimal dense coding has the same thermal behavior for both antiferromagnetic case and ferromagnetic case, when an external magnetic field is perpendicular to Ising orientation. From Fig. 3(b), the critical temperature T_{c_2} increases with the increasing of external magnetic field, while if $B \geq 1.4J$, no matter what temperature is, χ is always less than 1,which means that the thermal entangled states are not valid for optimal dense coding.

5 Concluding Remarks

In conclusion, the entanglement and the optimal dense coding of a two-qubit Ising system in the presence of arbitrary direction external magnetic fields have been studied. The effects of the direction of an external magnetic field and of temperature on the entanglement and the dense coding capacity have been analyzed. It has shown that the entanglement behavior and the maximal dense coding capacity of the system can be changed through turning on the angle between the direction of an external magnetic field and Ising orientation without manipulating the other parameters. The antiferromagnetic case and ferromagnetic case exhibit different entanglement behavior and present different optimal dense coding capacity. At zero temperature, for the antiferromagnetic case, in the region of $J > 0.5B$, the entanglement decreases with the increasing of θ and reaches minimum at $\theta = \pi/2$. The quantum phase transition has taken place at critical point $\theta = 0$. In contrast to antiferromagnetic case, if and only if an external magnetic field is perpendicular to Ising orientation, the maximal entanglement can be obtained for ferromagnetic case. At nonzero temperature, the concurrence decreases monotonously. The critical temperature T_{c_1}, beyond which the concurrence vanishes, increases with the increasing of θ, and reaches maximum at $\theta = \pi/2$ for both antiferromagnetic case and ferromagnetic case. The optimal dense coding capacity exhibits analogous characteristic to entanglement, no matter what temperature is for both antiferromagnetic case and ferromagnetic case. There is a region for θ, in which the optimal dense coding is feasible through the quantum channel. The critical temperature T_{c_2}, crossing which the optimal dense coding is not valid, reaches maximum at $\theta = \pi/2$ as the same as T_{c_1}.

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